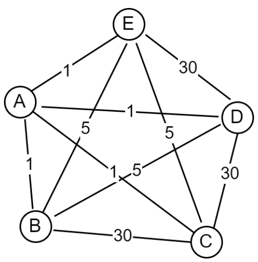
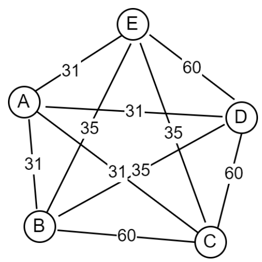
29.2. 2024

Inapproximability of general TSP

General to Metric TSP



$$\rightarrow$$
$$M = \max_{i,j} d(i,j)$$
$$d'(i,j) = d(i,j) + M$$



$$\begin{aligned}
\text{2-APX}(G,d) &\neq \text{2-APX}(G,d') \\
\text{2-APX}(G,d) &= c \leq 2OPT \\
\text{2-APX}(G,d') &= c' \leq 2OPT' = 2OPT + 2nM \\
c' &\leq 2OPT + 2nM - nM \text{ (Remove added weights)} \\
\frac{c'}{c} &= \frac{2OPT + nM}{2OPT} = 1 + \frac{nM}{OPT} \rightarrow \text{Very bad!}
\end{aligned}$$

Proof

Theorem:
If $P \neq NP$, then for any $S \geq 1$, there is no S-approximation algorithm for the general TSP.

Proof by contradiction:
Assume that there is a S-approximation algorithm for the general TSP. Then, we can solve the Hamiltonian cycle problem in polynomial time. But the Hamiltonian cycle problem is NP-complete, so statement $P = NP$ does not hold (CONTRADICTION).

$$\forall S \geq 1, \forall A \in: A \in S\text{-APX TSP} \rightarrow P = NP$$

$$A \in S\text{-APX TSP} \rightarrow A \text{ solves } HAM \rightarrow HAM \in \text{NP-complete} \rightarrow P \neq NP$$

A* Algorithm

Informed graph search algorithm. A* selects the path minimizing:

$$f(n) = g(n) + h(n)$$

$g(n)$ - cost from the start node to node n

$h(n)$ - estimated cost from node n to the goal - **heuristic function**

Consistent heuristic

A* is guaranteed to find the optimal solution if h is consistent. It must hold:

$$\forall (x,y) \in E : h(x) \leq d(x,y) + h(y)$$

$h(n)$ cannot overestimate the cost to reach the goal.