25.4. 2024

Random Walk 3-SAT

For clause $C = (\alpha, \beta, \gamma)$ if assignment A does not satisfy C. By random choice one of three literals is fliped in the assigment by notion of random walk. There is $A*$ that satisfies C with $\alpha = T$, then probability of A becoming $A*$ is:

$$P(A \to A*) = P(e_i \to e_{i+1}) = \frac{1}{3}$$
$$P(A \to \text{ not } A*) = P(e_i \to e_{i-1}) = \frac{2}{3}$$

Lets define e_i as expected number of steps needed to satisfy n clauses if i are already satisfied. Then:

$$\begin{aligned} e_n &= 0 \\ e_0 &= 1 + e_1, \text{ one step } + e + 1 \\ e_i &= 1 + \frac{1}{3}e_{i+1} + \frac{2}{3}e_{e-1} \end{aligned}$$

Lets define $d_i = e_{i-1} - e_i$, then:

$$\begin{aligned} 3e_i &= 3 + e_{i+1} + 2e_{i-1} \\ 2(e_i - e_{i-1}) &= 3 + (e_{i+1} - e_i) \\ 2d_{i-1} &= 3 + d_i \\ d_i &= 2d_{i-1} - 3 \\ e_i &= e_{i+1} + d_i, \text{ from definition} \\ e_i &= e_{i+1} + 2d_{i-1} - 3 \\ &= e_{i+1} + 2(2di - 2 - 3) - 3 \\ &= e_{i+1} + 2(2(2di - 3 - 3) - 3) - 3 \end{aligned}$$

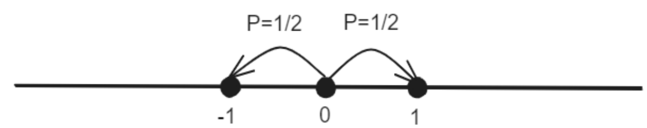
Therefore $e_0 \approx O(2^n)$, which is as bad as exhaustive search.

Expected distance from 0

Problem

Let Z be a random variable representing random step in the walk on the integer axis. Then:

$$P(Z = 1) = \frac{1}{2}, P(Z = -1) = \frac{1}{2}$$



What is the expected distance from 0 after t steps?

Solution

Let X_t be a random variable representing the position after t steps of random walk algorithm.

$$X_t = Z_1 + Z_2 + \cdots + Z_t$$

Square of the distance from 0 is X_t^2.

$$\begin{aligned} E[X_t^2] &= E[(Z_1 + \cdots + Z_t)^2] = E[(Z_1 + \cdots + Z_t)(Z_1 + \cdots + Z_t)] \\ &= E[\sum_{i=1}^{t} Z_i^2 + \sum_{1 \le i \le j \le t} 2Z_iZ_j] \\ &= \sum_{i=1}^{t} E[Z_i^2] + 2\sum_{1 \le i \le j \le t} E[Z_iZ_j] \end{aligned}$$

We need to compute $E[Z_i^2]$ and $[Z_iZ_j]$.

$$\begin{aligned} E[Z_i^2] &= \frac{1}{2}1^2 + \frac{1}{2}(-1)^2 = 1 \\ E[Z_iZ_j] &= \frac{1}{4}(1)(1) + \frac{1}{4}(1)(-1) + \frac{1}{4}(-1)(1) + \frac{1}{4}(-1)(-1) = 0 \end{aligned}$$

Then:

$$E[X_t^2] = \sum_{i=1}^{t} 1 + 2\sum_{1 \le i \le j \le t} 0 = t$$

By Cauchy-Schwarz inequality $E[X_t^2] \ge (E[X_t])^2$, then:

$$E[|X_t|] = \sqrt{t}$$

which is the expected distance.