Finding extremes in multivariate functions

- we use **partial derivates**
- compute derivative with respect to a **chosen variable**
  treat other variable as constants:

$$\frac{\partial f(a,b)}{\partial a} = \lim_{a \to x} \frac{f(x,b) - f(a,b)}{x - a}$$

$$\frac{\partial (x^2 + y^2 + xy)}{\partial x} = 2x + 0 + y$$

- **necessary condition:** if f has a **local extreme** at a particular point, **all partial derivatives** must be zero

Matrix multiplication

$A_{m \times n} \cdot B_{n \times k} = C_{m \times k}$

$C_{ij} = \sum_{u=1}^{n} A_{iu} B_{uj}$

$$\begin{bmatrix} 1 & 3 & 1 \\ 1 & 1 & 2 \\ 2 & 3 & 4 \end{bmatrix}_{3\times 3} \cdot \begin{bmatrix} 2 & 9 \\ 0 & -2 \\ -1 & 2 \end{bmatrix}_{3\times 2} = \begin{bmatrix} 1 & 5 \\ 0 & 11 \\ 0 & 20 \end{bmatrix}_{3\times 2}$$

Inverse matrices

$A_{m\times n} \cdot B_{n\times k} = C_{m\times k}$

$C_{ij} = \sum_{u=1}^{n} A_{iu}B_{uj}$

$$\begin{bmatrix} 1 & 3 & 1 \\ 1 & 1 & 2 \\ 2 & 3 & 4 \end{bmatrix}_{3\times 3} \cdot \underbrace{\begin{bmatrix} 2 & 9 & -5 \\ 0 & -2 & 1 \\ -1 & -3 & 2 \end{bmatrix}_{3\times 3}}_{\text{inverse matrix}} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3}}_{\text{identity matrix}}$$

$A \cdot A^{-1} = I$

$(AB)^{-1} = B^{-1} \cdot A^{-1}$

Transposed matrices

$A_{n\times m} = (a_{ij})_{n\times m}$

$(A^T)_{m\times n} = (a_{ji})_{m\times n}$

$$\begin{bmatrix} 1 & 3 & 1 \\ 1 & 1 & 2 \\ 2 & 3 & 4 \end{bmatrix}_{3\times 3} \cdot \begin{bmatrix} 2 & 9 \\ 0 & -2 \\ -1 & 2 \end{bmatrix}_{3\times 2} = \begin{bmatrix} 1 & 5 \\ 0 & 11 \\ 0 & 20 \end{bmatrix}_{3\times 2}$$

$$\begin{bmatrix} 2 & 0 & -1 \\ 9 & -2 & -3 \end{bmatrix}_{2\times 3} \cdot \begin{bmatrix} 1 & 1 & 2 \\ 3 & 1 & 3 \\ 1 & 2 & 4 \end{bmatrix}_{3\times 3} = \begin{bmatrix} 1 & 0 & 0 \\ 5 & 11 & 20 \end{bmatrix}_{2\times 3}$$

$(A \cdot B)^T = B^T \cdot A^T$

Gradient

Consider function $f : R_{m \times n} \to R$

$$\nabla_A f(A) = \begin{bmatrix} \frac{\partial f(A)}{\partial a_{11}} & \frac{\partial f(A)}{\partial a_{12}} & \cdots & \frac{\partial f(A)}{\partial a_{1n}} \\ \frac{\partial f(A)}{\partial a_{21}} & \frac{\partial f(A)}{\partial a_{22}} & \cdots & \frac{\partial f(A)}{\partial a_{2n}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f(A)}{\partial a_{m1}} & \frac{\partial f(A)}{\partial a_{m2}} & \cdots & \frac{\partial f(A)}{\partial a_{mn}} \end{bmatrix}$$

$(AB)^{-1} = B^{-1}A^{-1}$

$(AB)^T = B^T A^T$

Vector x: matrix with a single **column**

Dot product (skalárny súčin): $\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i = x^T y$

Matrix trace (stopa matice): $\mathrm{tr} A_{n \times n} = \sum_{i=1}^{n} A_{ii}$

$\mathrm{tr}(A + B) = \mathrm{tr} A + \mathrm{tr} B$

$\mathrm{tr} A^T = \mathrm{tr} A$

$\nabla_A \mathrm{tr}(BA) = B^T$

$\nabla_A \mathrm{tr}(A^T BAC) = B^T AC^T + BAC$