On Multiplicity of Permutation Graphs

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Abstract

In this article we deal with labeled and unlabeled permutation graphs. Let π be a permutation of elements $\{1, 2, \ldots, n\}$. *Labeled graph* of π is a graph where vertices are labeled by numbers $\{1, \ldots, n\}$ and edges correspond to inversions in π. *Unlabeled graph* of π is a graph without vertex labels which is isomorphic to labeled graph of π. Several permutations can be represented by the same unlabeled graph. Number of such permutation is called *multiplicity* of the graph.

We give characterization of the family of labeled permutation graphs. We use this characterization and symmetry to show that multiplicity of unlabeled graphs G and $\overline{G}$ is the same (*). It is also possible to derive other results in multiplicity of special families of graphs using our characterization.

This work was inspired by lectures led by prof. Frank Harary at the Summer School at Jyväskyla University in Finland in 1996. The problem of multiplicity of permutation graphs was stated by him and he also stated (*) as a conjecture. Unfortunately we was not able to find motivation for multiplicity of permutation graphs (although it seems that there is some) and we would be glad to hear about such motivation.

1 Labeled Permutation Graphs

Definition 1 *Let π be a permutation on set $\{1, 2, \ldots, n\}$. Assume $V = \{1, 2, \ldots, n\}$. Non-oriented graph $G_\pi = (V, E)$ is a labeled permutation graph of permutation π if for all $u, v \in V, u < v$ is $(\{u, v\} \in E) \Leftrightarrow (\pi(u) > \pi(v))$.*

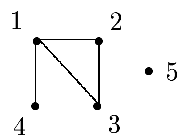

Figure 1: A labeled permutation graph for $\pi = (4\ 2\ 1\ 3\ 5)$

Notation. We assume that the node set for any labeled graph $V = \{1, 2, \ldots, n\}$ throughout this article. We will denote S_n the set of permutations on set $\{1, 2, \ldots, n\}$.

At first we will introduce the presentation of labeled permutation graphs which is helpful to illustrate deriving of some results throughout this article.

Consider the permutation graph G_π. It is possible to assign to each node i of graph G coordinates $(i, \pi(i))$. This way it is possible to "draw" the graph into the plane (see figure 1). We will say that the graph is *drawn in the plane.*

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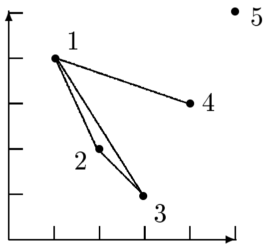

Figure 2: G_π drawn in the plane for $\pi = (4\ 2\ 1\ 3\ 5)$

Notice that for each permutation graph "drawn in the plane" two nodes u, v (suppose $u < v$) are adjacent if and only if node v is right and down in relation to the u (implies from definition).

2 Labeled Permutation Graphs and Permutations

It is clear from the definition 1 that for each permutation it is possible to construct its unique labeled permutation graph.

Two questions we will deal with in this section:

1. Does the labeled permutation graph represent one or more permutations?

2. Does each labeled graph represent any permutation?

Definition 2 *Let $G = (V, E)$ be a labeled graph. For $u, v \in V$ is $u \prec_G v$ if $u < v$ and $\{u, v\} \notin E$ (u, v are not adjacent in G) or $u > v$ and $\{u, v\} \in E$ (u, v are adjacent in G).*

Notice that for all u, v ($u \neq v$) is $u \prec_G v$ or $v \prec_G u$.

Lemma 3 *The labeled graph $G = (V, E)$ is a labeled permutation graph if and only if the relation $\prec_G$ defines an ordering on set V.*

Proof.

($\Rightarrow$) Let G be a labeled permutation graph. It is easy to see, that the $\prec_G$ is an ordering on V (notice that $u \prec_G v$ if and only if $\pi(u) < \pi(v)$).

($\Leftarrow$) Let $G = (V, E)$ be any labeled graph for which $\prec_G$ is an ordering. Let $|V| = n$. We will show that there exists a permutation π for which the graph G_π is isomorphic to G.

Because the $\prec_G$ is an ordering it is possible to find a sequence $a_1, a_2, \ldots, a_n$ ($\{a_1, a_2, \ldots, a_n\} = V$) where $a_i \prec_G a_{i+1}$ for $i = 1, 2, \ldots, n-1$.

Let π be a permutation such that $\pi(a_i) = i$ for $i = 1, 2, \ldots, n$. Let $u, v \in V$, $u < v$. If u, v are adjacent in G then $u \prec_G v$ and therefore $\pi(v) < \pi(u)$ and u, v are adjacent in G_π, too. Similarly if u, v are not adjacent in G then $u \prec_G v$ and therefore $\pi(u) < \pi(v)$ and u, v are not adjacent in G_π, too. Therefore G_π is isomorphic to G. $\square$

As a consequence of the proof of lemma 3 we are able to answer the first question in the following theorem.

Theorem 4 *A labeled graph $G = (V, E)$ represents at most one permutation.*

Proof. Let G be a labeled permutation graph. It can be easily shown (analogously to proof of lemma 3) that $u \prec_G v$ if and only if $\pi(u) < \pi(v)$. Therefore, π is unambiguously determined. □

Lemma 3 gives negative answer to the second question. Figure 3 shows the labeled graph G for which the relation $\prec_G$ is not an ordering therefore (according to lemma 3) it cannot be a labeled permutation graph.

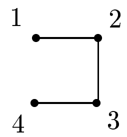

Figure 3: $\prec_G$ is not a complete ordering.

We are now able to determine a necessary and sufficient condition to decide whether a labeled graph is a permutation graph or not.

Definition 5 *We say that the labeled graph $G = (V, E)$ is partially transitive if for each $u, v, w \in V$, $u < v < w$, if u, v and v, w are both adjacent then u, w are adjacent too.*

Theorem 6 *A labeled graph $G = (V, E)$ is a permutation graph if and only if G and its complement $\overline{G}$ are both partially transitive.*

Proof.
($\Rightarrow$) Let π be a permutation and G_π be its permutation graph. We will show that G_π and $\overline{G_\pi}$ are both partially transitive.

Take three nodes u, v, w, where $u < v < w$. If u, v and v, w are both adjacent then (from definition of permutation graph) we have $\pi(u) > \pi(v)$ and $\pi(v) > \pi(w)$ and therefore $\pi(u) > \pi(w)$ and u, w are adjacent. Similarly if u, v and v, w are both not adjacent we have $\pi(u) < \pi(v)$ and $\pi(v) < \pi(w)$ and therefore $\pi(u) < \pi(w)$ and u, w are not adjacent, too. Therefore both G_π and $\overline{G_\pi}$ are partially transitive.

($\Leftarrow$) Let $G = (V, E)$ be a labeled graph satisfying conditions of the theorem. We will show that there exists a permutation π where G_π is isomorphic to G. According to lemma 3 it is sufficient to prove that $\prec_G$ is an ordering on V (it means $\prec_G$ is antisymmetric and transitive).

Antisymmetricity follows from the definition 2.

For the transitivity: Take three nodes u, v, w for which $u \prec_G v$ and $v \prec_G w$. There are six posibilities to discuss:

- $u < v < w$. Then u and v are adjacent and v and w are adjacent, too. From partial transitivity of G follows that u and w are adjacent and therefore $u \prec_G w$.
- $w < u < v$. Then u and v are adjacent. For the contradiction suppose that u and w are adjacent. From partial transitivity of G v and w are adjacent and therefore $w \prec_G v$ which contradicts to $v \prec_G w$. Therefore u and w must not be adjacent and $u \prec_G w$.
- $v < w < u$. This case can be proved in the same way as the case $w < u < v$.
- $w < v < u$, $v < u < w$, $u < w < v$. These cases can be proved in the same way using the partial transitivity of $\overline{G}$ instead of partial transitivity of G.

□

3 Symmetry on Permutation Graphs

We will now use the symmetry and the "drawn in the plane" presentation of the labeled permutation graph G_π of permutation π to derive some properties of permutation graphs.

At first we will introduce unlabeled permutation graphs which are derived from labeled permutation graphs.

Definition 7 *Unlabeled permutation graph G is unlabeled graph such that there is a labeled permutation graph G_π isomorphic to G. We will use notation G_π^- for unlabeled permutation graph which is isomorphic to G_π.*

Lemma 8 *Let G_π be a labeled permutation graph of permutation $\pi \in S_n$. Let π' be a permutation given by formula $\pi'(i) = \pi(n+1-i)$ for $i = 1, \ldots, n$. Then unlabeled permutation graph $G_{\pi'}^-$ is isomorphic to $\overline{G_\pi^-}$.*

Proof. Take the "drawn in the plane" presentation of graph G_π. The permutation π' is obtained by symmetry on $x = \frac{n+1}{2}$ axis. Consider the labeled permutation graph $G_{\pi'}$ ("drawn in the plane").

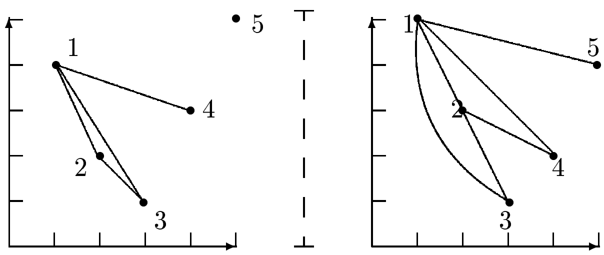

Figure 4: Graphs G_π and $G_{\pi'}$

Let $u, v \in V$ be vertices in the graph G_π and u', v' be their images in considered symmetry.

Let u, v are adjacent in G_π. Then in the "drawn in tha plane" presentation v is right and down to u. In the cosidered symmetry the v' will now left and down to u' so they are not adjacent in $G_{\pi'}$. Similarly if u, v are not adjacent in G_π then u', v' are adjacent in $G_{\pi'}$. $\square$

Notice that the proof of the lemma 8 could be done by using the relation $\prec_G$ on permutation graphs instead of "drawn in the plane" presentation and symmetry, too.

Using of other symmetries on the "drawn in the plane" presentation we can derive following two statements.

Lemma 9 *Let G_π be a labeled permutation graph of permutation $\pi \in S_n$. Let π' be a permutation given by formula $\pi'(i) = n + 1 - \pi(i)$ for $i = 1, \ldots, n$ ($\pi' = \pi^{-1}$ respectively). Then unlabeled permutation graph $G_{\pi'}^-$ is isomorphic to $\overline{G_\pi^-}$ (G_π^- respectively).*

Proof. Similarly to the proof of lemma 8 using symetry on $y = \frac{n+1}{2}$ axis ($y = x$ axis respectively) $\square$

4 Multiplicity

According to the theorem 4 the labeled permutation graph G_π represents at most one permutation. It is not valid for unlabeled permutation graphs. For example for each permutation π graphs G_π^- and $G_{\pi^{-1}}^-$ are isomorphic.

Definition 10 *The multiplicity of unlabeled graph G is number of permutations π such that graph G is isomorphic to the labeled permutation graph G_π.*

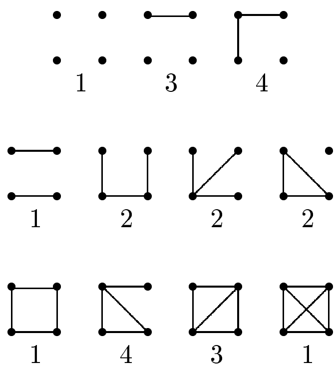

Figure 5: Multiplicity of 4-node graphs

Notice that the graph G is unlabeled permutation graph if and only if its multiplicity is at least 1.

Our considerations about the symmetry give us possibility to derive following theorem.

Theorem 11 *Multiplicity of unlabeled graphs G and $\overline{G}$ is the same.*

Proof. Let S be the set of permutations represented by graph G (with n nodes) and S' be the set of permutations represented by graph $\overline{G}$. Consider the projection $f : S_n \to S_n$ such that $f(\pi) = \pi'$ where $\pi'(i) = \pi(n+1-i)$ for $i = 1, 2, \ldots, n$. Notice that $f^{-1} = f$

According to lemma 8 $f(S) \subseteq S'$ and $f^{-1}(S') = f(S') \subseteq S$. Therefore $|S| = |S'|$.
□