The Asymptotic Cheat Sheet

Asymptotic notation consists of six funny symbols used to describe the relative growth rates of functions. These six symbols are defined in the table below.

$f = \Theta(g)$	f grows at the same rate as g	There exists an n_0 and constants $c_1, c_2 > 0$ such that for all $n > n_0$, $c_1 g(n) \leq \|f(n)\| \leq c_2 g(n)$.
$f = O(g)$	f grows no faster than g	There exists an n_0 and a constant $c > 0$ such that for all $n > n_0$, $\|f(n)\| \leq cg(n)$.
$f = \Omega(g)$	f grows at least as fast as g	There exists an n_0 and a constant $c > 0$ such that for all $n > n_0$, $cg(n) \leq \|f(n)\|$.
$f = o(g)$	f grows slower than g	For all $c > 0$, there exists an n_0 such that for all $n > n_0$, $\|f(n)\| \leq cg(n)$.
$f = \omega(g)$	f grows faster than g	For all $c > 0$, there exists an n_0 such that for all $n > n_0$, $cg(n) \leq \|f(n)\|$.
$f \sim g$	f/g approaches 1	$\lim_{n\to\infty} f(n)/g(n) = 1$

The $\sim$ and Θ notations are confusingly similar; qualitatively, functions related by $\sim$ must be even more nearly alike then functions related by Θ. The ω notation makes the table nice and symmetric, but is almost never used in practice. Some asymptotic relationships between functions imply other relationships. Some examples are listed below.

$$
\begin{array}{rcl}
f = O(g) \text{ and } f = \Omega(g) & \Leftrightarrow & f = \Theta(g) \\
f = O(g) & \Leftrightarrow & g = \Omega(f) \\
f = o(g) & \Leftrightarrow & g = \omega(f)
\end{array}
\qquad
\begin{array}{rcl}
f = o(g) & \Rightarrow & f = O(g) \\
f = \omega(g) & \Rightarrow & f = \Omega(g) \\
f \sim g & \Rightarrow & f = \Theta(g)
\end{array}
$$

Limits

The definitions of the various asymptotic notations are closely related to the definition of a limit. As a result, $\lim_{n\to\infty} f(n)/g(n)$ reveals a lot about the asymptotic relationship between f and g, provided the limit exists. The table below translates facts about the limit of f/g into facts about the asymptotic relationship between f and g.

$\lim_{n\to\infty} f(n)/g(n) \neq 0, \infty$	$\Rightarrow$	$f = \Theta(g)$	$\lim_{n\to\infty} f(n)/g(n) = 1$	$\Rightarrow$	$f \sim g$
$\lim_{n\to\infty} f(n)/g(n) \neq \infty$	$\Rightarrow$	$f = O(g)$	$\lim_{n\to\infty} f(n)/g(n) = 0$	$\Rightarrow$	$f = o(g)$
$\lim_{n\to\infty} f(n)/g(n) \neq 0$	$\Rightarrow$	$f = \Omega(g)$	$\lim_{n\to\infty} f(n)/g(n) = \infty$	$\Rightarrow$	$f = \omega(g)$

Therefore, skill with limits can be helpful in working out asymptotic relationships. In particular, recall L'Hospital's Rule:

$$\text{If } \lim_{n\to\infty} f(n) = \infty \text{ and } \lim_{n\to\infty} g(n) = \infty, \text{ then } \lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{f'(n)}{g'(n)}.$$

Every computer scientist knows two rules of thumb about asymptotics: logarithms grow more slowly than polynomials and polynomials grow more slowly than exponentials. We'll prove these facts using limits.

Theorem. *For all $\alpha, k > 0$:*

$$(\ln n)^k = o(n^\alpha) \tag{1}$$
$$n^k = o((1+\alpha)^n) \tag{2}$$

Proof.

$$\lim_{n\to\infty} \frac{(\ln n)^k}{n^\alpha} = \left(\lim_{n\to\infty} \frac{\ln n}{n^{\alpha/k}}\right)^k \stackrel{*}{=} \left(\lim_{n\to\infty} \frac{1/n}{(\alpha/k)n^{\alpha/k-1}}\right)^k = \left(\lim_{n\to\infty} \frac{1}{(\alpha/k)n^{\alpha/k}}\right)^k = 0$$

$$\lim_{n\to\infty} \frac{n^k}{(1+\alpha)^n} = \left(\lim_{n\to\infty} \frac{n}{(1+\alpha)^{n/k}}\right)^k \stackrel{*}{=} \left(\lim_{n\to\infty} \frac{1}{(n/k)\cdot(1+\alpha)^{n/k-1}}\right)^k = 0$$

The starred equalities follow from L'Hospital's Rule. $\square$